THE THEORY OF A FREE-FALL EQUIVALENCE-PRINCIPLE EXPERIMENT IN A DRAG-FREE SATELLITE

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The simplest and most straight forward satellite test of the Universality of Free Fall (EP) is to use two concentric proof masses, a sphere and a spherical shell, in a Drag-Free satellite. It is possible to use two proof masses if, prior to beginning the experiment, the inner sphere is centered in the cavity by the drag-free control system and the outer shell is centered by an electric suspension system. After the experiment begins the electric suspension is turned off and the two proof mass execute free fall in space.

While this configuration is simple in principle, it suffers from two problems which make such an experiment essentially impossible: 1) the two proof masses will always travel in separate orbits and the semimajor axes must be identical to within an error of $x_0 = -f_x / 3n^2$ where x_0 is the semimajor-axis error, f_x is the violation of the EP, and n is the mean orbit angular velocity; 2) the position differences of the proof masses only grows as t in response to an EP violation and not as t^2. As an example, detecting an EP violation of 10^{-19} g (10^{-18} m/sec^2) would require that the two orbits be identical to within about 3×10^{-13} meters. The second problem is important because the major disturbance source grows as $t^{3/2}$ so that the accuracy of the experiment actually worsens with time as $t^{1/2}$ if the separation only grows as t.

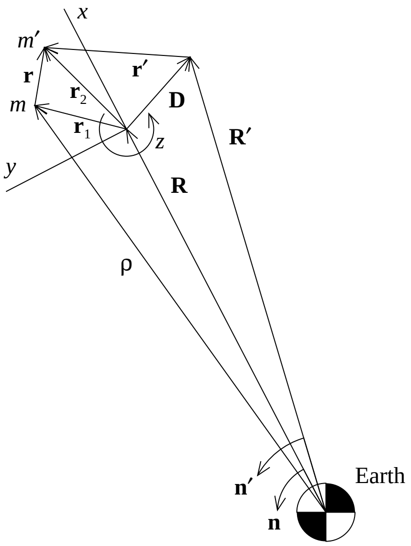

Figure 1. Coordinate System

These two problems can be solved by altering the orbit dynamics of the two proof masses by placing two large gravity-gradient cancellation masses on each side of the proof masses centered along a line perpendicular to the orbit plane (See Figure 1 in Reference 1). This paper will describe the theory behind this approach.

In the coordinate system defined by Figure 1 where x is in the radial direction, y is parallel to the orbit velocity vector, and z is perpendicular to the orbit plane, the relative equations of motion of the two proof masses with respect to a circular orbit are [1]

$$\ddot{\mathbf{r}} = -n^2(\mathbf{1} - 3\hat{\mathbf{R}}\hat{\mathbf{R}}) \cdot \mathbf{r} + \mathbf{f} + \mathbf{f}_d + \Delta\mathbf{f}_{ns} \qquad \mathbf{f} = -\Delta\mu\mathbf{R}/R^3 \qquad \mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}$$

where $\Delta\mu$ is the difference in the gravitational constants of the two proof masses used to model any violation of the universality of free fall. Separating out the gravity-gradients of the spacecraft explicitly from the disturbances, $\mathbf{f}_d$, we have

$$\begin{bmatrix} \ddot{x} - n^2x - 2n\dot{y} \\ 2n\dot{x} + \ddot{y} - n^2y \\ \ddot{z} \end{bmatrix} = \left\{ \begin{bmatrix} 2n^2 & 0 & 0 \\ 0 & -n^2 & 0 \\ 0 & 0 & -n^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(g_{11} + g_{22}) & 0 & 0 \\ 0 & \frac{1}{2}(g_{11} + g_{22}) & 0 \\ 0 & 0 & g_{33} \end{bmatrix} + \right.$$

$$\left. \begin{bmatrix} \frac{1}{2}(g_{11} - g_{22})c2 - g_{12}s2 & \frac{1}{2}(g_{11} - g_{22})s2 + g_{12}c2 & g_{13}c - g_{23}s \\ \frac{1}{2}(g_{11} - g_{22})s2 + g_{12}c2 & -\frac{1}{2}(g_{11} - g_{22})c2 + g_{12}s2 & g_{13}s + g_{23}c \\ g_{13}c - g_{23}s & g_{13}s + g_{23}c & 0 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} f_x + f_{dx} + \Delta f_{nsx} \\ f_{dy} + \Delta f_{nsy} \\ f_{dz} + \Delta f_{nsz} \end{bmatrix} \tag{1}$$

The terms on the left are acceleration in a rotating frame and the terms on the right arise from the gravity gradient tensors of the earth and the satellite. c and s are the sine and cosine of the satellite rotation angle, and s2 and c2 are the sine and cosine of twice the angle. g_{ij} are the elements of the gravity gradient tensor from the masses in the satellite. g_{11}, g_{22}, and g_{33} are assumed to be large, of the order of $n^2 \approx 10^{-6}/\mathrm{sec}^2$, while g_{12}, g_{13}, g_{23} , and g_{11} - g_{22} are assumed to be small, of the order of 10^{-8} to $10^{-12}/\mathrm{sec}^2$. The time-varying sine and cosine terms arise because the g_{ij} are transformed from the spacecraft into the orbit frame of Figure 1. f_x is a disturbance which models the violation of the EP, f_{dxyz} represents the disturbance force errors from the satellite, and Δf_{nsxyz} the difference in the non-spherical gravitation terms of the earth.

In the usual analysis, all of the satellite gravity gradient terms are assumed to be small and Eq. 1 reduces to the Euler-Hill equations shown on the left side of Eq. 2. When the DC-Cancellation masses are chosen such that $\frac{1}{2}(g_{11} + g_{22}) = -3n^2$ and $g_{11} - g_{22} = 0$, Eq. 1 is transformed into the modified Euler-Hill equations show on the right of Eq. 2.

$$\begin{array}{c} \ddot{x} - 3n^2 - 2n\dot{y} = f_x \\ 2n\dot{x} + \ddot{y} = 0 \\ \ddot{z} + n^2z = 0 \end{array} \xrightarrow[\text{DC Cancellation}]{} \begin{array}{c} \ddot{x} - 2n\dot{y} = f_x \\ 2n\dot{x} + \ddot{y} + 3n^2y = 0 \\ \ddot{z} - 5n^2z = 0 \end{array} \,. \tag{2}$$

The solutions with an initial x_0 and an EP violation, f_x, are then transformed as

$$\begin{aligned} x &= x_0 + (f_x + 3n^2 x_0)(1-cnt)/n^2 \\ y &= -2(f_x + 3n^2 x_0)(nt - snt)/n^2 \end{aligned} \xrightarrow{\textit{DC Cancellation}} \begin{aligned} x &= x_0 + \frac{3}{14} f_x t^2 + \frac{4f_x}{49n^2}(1 - c\sqrt{7}nt) \\ y &= -\frac{2f_x}{7n}t + \frac{2f_x}{7\sqrt{7}n^2} s\sqrt{7}nt \end{aligned} \qquad (3)$$

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Notice that the solution of Eq. 2 without DC Cancellation on the left side of Eq. 3 shows that the difference between the proof masses grows in the y direction as t and that the combination $f_x + 3n^2 x_0$ confounds the measurement of the EP violation, f_x, and the semimajor-axis error, x_0. In the ideal case of perfect cancellation, the solution on the right of Eq. 3 shows that the combination $f_x + 3n^2 x_0$ is reduced to f_x alone and that the deviation between the proof masses grows as t^2 in the x direction, i.e. radially downward. Because of the term 3 / 14, the experiment with cancellation masses is equivalent to a very tall drop tower with a constant acceleration in the vertical direction equal to 3 / 7 g.

Because of imperfect cancellation, the right side of Eq. 3 is only valid for a finite time. Table 1 shows the equivalent drop-tower distances for various cancellation accuracies. The cancellation accuracy is defined to be $k^2/3n^2$ where $k^2 = 3n^2 + (g_{11} + g_{22})/2$.

Drop Time (sec) t	Cancellation Accuracy	Distance (km) $\frac{3}{14}gt^2$	Distance (AU)
10^4	10^{-2}	210,000	0.0015
10^5	10^{-4}	21,000,000	0.15
10^6	10^{-6}	2100,000,000	15

Table 1. Equivalent Drop-Tower Parameters with Various DC-Cancellation Accuracies

A cancellation accuracy of 10^{-2} (i.e. a residual gravity gradient of 3×10^{-8} /sec^2) was obtained in the Disco Drag-Free satellite which flew in 1972 [2]. This guarantees that this level of cancellation is achievable.

Practical problems are: 1) Cancellation accuracy, 2) Disturbing forces from the satellite, 3) Initial conditions, 4) Instability of the z-axis, and 5) Measuring the positions of the sphere and shell. These are discussed in detail in Reference 1 which shows how to solve these problems so that the resulting experiment error lies between 10^{-18} and 10^{-20} g. This result is dominated by the disturbance noise and not by the measurement noise which can be as large as 10^{-9} m/Hz$^{1/2}$.

References

1. gr-qc/0005073 or www.dragfreesatellite.com/DC_Can3.pdf
2. www.dragfreesatellite.com